

THE SYMPLECTIC GEOMETRY OF POLYGONS IN THE 3-SPHERE

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ABSTRACT. We study the symplectic geometry of the moduli spaces $M_r = M_r(\mathbb{S}^3)$ of closed n -gons with fixed side-lengths in the 3-sphere. We prove that these moduli spaces have symplectic structures obtained by reduction of the fusion product of n conjugacy classes in $SU(2)$, denoted C_r^n , by the diagonal conjugation action of $SU(2)$. Here C_r^n is a quasi-Hamiltonian $SU(2)$ -space. An integrable Hamiltonian system is constructed on M_r in which the Hamiltonian flows are given by bending polygons along a maximal collection of nonintersecting diagonals. Finally, we show the symplectic structure on M_r relates to the symplectic structure obtained from gauge-theoretic description of M_r . The results of this paper are analogues for the 3-sphere of results obtained for $M_r(\mathbb{H}^3)$, the moduli space of n -gons with fixed side-lengths in hyperbolic 3-space [KMT], and for $M_r(\mathbb{E}^3)$, the moduli space of n -gons with fixed side-lengths in \mathbb{E}^3 [KM1].

1. INTRODUCTION

In this paper we study the symplectic geometry of the space of polygons in \mathbb{S}^3 with fixed side-lengths modulo the group of isometries. We denote this moduli space by $M_r = M_r(\mathbb{S}^3)$. This paper is continuation of [KM1] and [KMT], which studied the polygonal linkages in Euclidean 3-space and hyperbolic 3-space, respectively.

An (open) n -gon P in \mathbb{S}^3 is an ordered $(n+1)$ -tuple (x_1, \dots, x_{n+1}) of points in $\mathbb{S}^3 \subset \mathbb{C}^2$ called the vertices. We join the vertex x_i to the vertex x_{i+1} by the unique geodesic segment e_i , called the i -th edge (here we must make the restriction x_i and x_{i+1} are not antipodal points). We let Pol_n denote the space of n -gons in \mathbb{S}^3 . An n -gon is said to be closed if $x_{n+1} = x_1$. We let $CPol_n$ denote the space of closed n -gons. The group $G = SU(2) \times SU(2)$ acting on \mathbb{S}^3 by $g \cdot x = g_1 x g_2^{-1}$, $x \in \mathbb{S}^3$, $g = (g_1, g_2) \in G$, is the group of isometries of \mathbb{S}^3 . Two n -gons $P = (x_1, \dots, x_{n+1})$ and $P' = (x'_1, \dots, x'_{n+1})$ are equivalent if there exists $g \in G$ such that $g \cdot P = P'$, that is $g \cdot x_i = x'_i$, for all $1 \leq i \leq n+1$.

Let $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ be an n -tuple of positive numbers with $r_i < \pi$ for $1 \leq i \leq n$. We denote by \tilde{N}_r the space of open n -gons in which the side e_i has fixed length $d(x_i, x_{i+1}) = r_i$. We then let $\tilde{M}_r = \tilde{N}_r \cap CPol_n$, $N_r = \tilde{N}_r/G$, and $M_r = \tilde{M}_r/G$. This paper examines the symplectic geometry of the space M_r .

We have $G = SU(2) \times SU(2)$, K is the diagonal subgroup in G , and $P = G/K$ which we identify with $SU(2)$. We equip G, K, P with the quasi-Poisson structures associated to the standard Manin pair $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g} = \{(x, y) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)\}$ and $\mathfrak{k} = \{(x, x) \in \mathfrak{g} : x \in \mathfrak{su}(2)\}$.

The main theorem of this paper is:

Date: February 1, 2008.

[†] Research partially supported by NSF grant DMS-98-03518.

Theorem 1.1. *The space M_r is a symplectic manifold with the symplectic structure obtained from reduction of the fusion product of n conjugacy classes in $SU(2)$, $C_{r_1} \otimes \cdots \otimes C_{r_n}$, by the diagonal dressing action (conjugation) of the quasi-Poisson Lie group K .*

We are also interested in finding an integrable system on M_r . We denote by d_{ij} a geodesic connecting the vertices x_i and x_j (we always assume $i < j$), which we call a diagonal. Let ℓ_{ij} be the length of the diagonal d_{ij} . Then ℓ_{ij} is a continuous function on M_r , but it is not smooth when either $\ell_{ij} = 0$ or $\ell_{ij} = \pi$. If d_{ij} and d_{km} are nonintersecting diagonals, then

$$\{\ell_{ij}, \ell_{km}\} = 0.$$

By considering a maximal collection of nonintersecting diagonals, we obtain $\frac{1}{2}\dim(M_r)$ Poisson commuting Hamiltonians.

The Hamiltonian flow Ψ_{ij}^t associated to a ℓ_{ij} has the following nice description. Separate the polygon into two pieces via the diagonal d_{ij} , the Hamiltonian flow is given by leaving one piece fixed while rotating the other piece about the diagonal at constant angular velocity 1. The flow Ψ_{ij}^t is called the “bending flow” along the diagonal d_{ij} .

The paper is organized as follows:

In section 2, we give background material for Manin pairs and quasi-Poisson Lie groups.

In section 3, we define a symplectic structure on M_r by quasi-Hamiltonian reduction on the fusion product of conjugacy classes.

In section 4, we study the Hamiltonians ℓ_{ij} and their associated Hamiltonian flows.

In section 5, we study the action of the pure braid group on M_r given by the time 1 Hamiltonian flows of a certain family of functions.

In section 6, we relate the symplectic form on M_r to symplectic form given on the relative character varieties on n -punctured 2-spheres.

We note that the moduli spaces of polygons in the spaces of constant curvature give examples of completely integrable systems obtained from the theory of Manin pairs associated to a compact simple Lie group [AMM2]. The Manin pairs corresponding to the various moduli spaces are:

- $(\mathfrak{su}(2) \ltimes \mathfrak{su}(2)^*, \mathfrak{su}(2))$ for polygons in the zero curvature space (Lie-Poisson theory);
- $(\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}(2)^{\mathbb{C}}, \mathfrak{su}(2))$ for polygons in negative curvature space (Poisson-Lie theory);
- $(\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{su}(2))$ for polygons in positive curvature space (quasi-Poisson Lie theory).

Acknowledgments

The author would like to thank John Millson for introducing him to the symplectic geometry of polygons and for numerous fruitful discussions. Thanks are due to Bill Goldman for many useful conversations. The author would also like to thank Eckhard Meinrenken bringing to his attention [AKS] and Proposition 2.8.

2. MANIN PAIRS AND QUASI-POISSON LIE GROUPS

2.1. quasi-Poisson Structures. In this section, we let K be any compact simple Lie group with Lie algebra denoted by \mathfrak{k} . Let $G = K \times K$ be the double of K with Lie algebra

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$. The Killing form on \mathfrak{l} , which we denote by (\cdot, \cdot) , defines a nondegenerate bilinear form $B(\cdot, \cdot)$ on \mathfrak{g} given by

$$B((X_1, X_2), (Y_1, Y_2)) = (X_1, Y_1) - (X_2, Y_2), \text{ for } (X_1, X_2), (Y_1, Y_2) \in \mathfrak{g}.$$

If we now let K denote the diagonal subgroup of G then its Lie algebra \mathfrak{k} is a maximal isotropic subalgebra of \mathfrak{g} . The pair $(\mathfrak{g}, \mathfrak{k})$ is a Manin pair. We will construct a quasi-Poisson Lie group structure on G associated to the Manin pair $(\mathfrak{g}, \mathfrak{k})$ which restricts to a (trivial) quasi-Poisson Lie group structure on K . For background on quasi-Poisson Lie groups, quasi-Poisson structures, Manin pairs, etc. we refer the reader to [AKS], [Le], [KS1], [KS2].

Let $\mathfrak{p} = \{(\frac{1}{2}X, -\frac{1}{2}X) \in \mathfrak{g}\}$ be the anti-diagonal in \mathfrak{g} . Then \mathfrak{p} is an isotropic complement of \mathfrak{k} . Note that \mathfrak{p} is not a Lie subalgebra of \mathfrak{g} ($[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$), so the triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is a Manin quasi-triple, rather than a Manin triple which arises in the theory of Poisson Lie groups. We call this triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ the standard Manin quasi-triple.

A Manin quasi-triple gives rise to a Lie quasi-bialgebra $(\mathfrak{k}, F, \varphi)$. We can identify \mathfrak{p} with \mathfrak{k}^* via the bilinear form of \mathfrak{g} . The cobracket on \mathfrak{k} is a map $F : \mathfrak{k} \rightarrow \mathfrak{k} \wedge \mathfrak{k}$ which is the transpose of the map from $\mathfrak{p} \wedge \mathfrak{p} \rightarrow \mathfrak{p}$, also denoted by F , defined by

$$F(\xi, \eta) = \rho_{\mathfrak{p}}[\xi, \eta], \quad \xi, \eta \in \mathfrak{p}.$$

We can also define the element $\varphi \in \wedge^3 \mathfrak{k}$ by the map $\mathfrak{p} \wedge \mathfrak{p} \rightarrow \mathfrak{k}$ given by

$$\varphi(\xi, \eta) = \rho_{\mathfrak{k}}[\xi, \eta], \quad \xi, \eta \in \mathfrak{p}.$$

For the Manin quasi triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ given above, we have $F = 0$ and $\varphi = \frac{1}{24} \sum_{ijk} f_{jk}^i e_i \wedge e_j \wedge e_k$, where $[e_j, e_k] = \sum_i f_{jk}^i e_i$.

We can also identify \mathfrak{g} with $\mathfrak{k} \oplus \mathfrak{k}^*$ via the bilinear form $B(\cdot, \cdot)$. The canonical r -matrix on \mathfrak{g} associated to the Manin quasi-triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is an element $r_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}$ defined by the map $r_{\mathfrak{g}} : \mathfrak{g}^* \rightarrow \mathfrak{g}$ given by $r_{\mathfrak{g}}(\xi, X) = (0, \xi)$ where $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. Let $\{e_i\}$ be an orthonormal basis of \mathfrak{k} and $\{\varepsilon^i\}$ be the dual basis in \mathfrak{k}^* , then

$$r_{\mathfrak{g}} = \sum_i e_i \otimes \varepsilon^i.$$

The multiplicative 2-tensor $w_G = dL_g r_{\mathfrak{g}} - dR_g r_{\mathfrak{g}}$ actually defines a bivector on G , since the symmetric part of $r_{\mathfrak{g}}$ is a multiple of the bilinear form $B(\cdot, \cdot)$ on \mathfrak{g} . $w_{\mathfrak{g}}$ gives us a quasi-Poisson Lie group structure on G . $w_{\mathfrak{g}}$ naturally restricts to the trivial bivector on the subgroup $K \subset G$. There is also a natural projection of $w_{\mathfrak{g}}$ to $G/K = P$, which can be identified with K , via the map $p : G \rightarrow P$ defined by $p(g_1, g_2) = g_1 g_2^{-1}$. The bivector w_P is given by

$$w_P = \frac{1}{2} \sum_i e_i^\lambda \wedge e_i^\rho.$$

Here e_i^λ (e_i^ρ) denotes the left-invariant (resp. right-invariant) vector field on P with value e_i at the identity. We will use this notation for vector fields on P throughout the rest of the paper. Note that w_P is not multiplicative, so P is not a quasi-Poisson Lie group. We will see that in the next section that P is the target space of a generalized moment map.

2.2. Moment map and reduction. The action of G on itself is by left multiplication induces an action of K on P , the dressing action, which is given by conjugation.

We denote by x_M the vector field, more generally the multivector field, on M induced by the action of K on M and $x \in \mathfrak{k}$ satisfying

$$(x_M f)(m) = \frac{d}{dt} \Big|_{t=0} f(\exp(-tx) \cdot m)$$

where $f \in C^\infty(M)$ and $m \in M$. This is a Lie algebra homomorphism, i.e. $[x_M, y_M] = [x, y]_M$ for $x, y \in \mathfrak{k}$.

We have the following definition of a quasi-Poisson action.

Definition 2.1. *Let (K, w_K, φ) be a connected quasi-Poisson Lie group acting on a manifold M with bivector w_M . The action of K on M is said to be a quasi-Poisson action if and only if*

- (i) $\frac{1}{2}[w_M, w_M] = \varphi_M$
- (ii) $\mathcal{L}_{x_M} w_M = -(F(x)_M)$

for all $x \in \mathfrak{k}$.

The dressing action of K on P is a quasi-Poisson action. There is also a notion of a generalized moment map associated to a quasi-Poisson action.

Definition 2.2. *A map $\mu : M \rightarrow P$, equivariant with respect to the action of K on M and the dressing action of K on P , is called a moment map for the action of K on (M, w_M) if, on any open subset of M ,*

$$w^\sharp(\mu^* \alpha_x) = x_M.$$

Here $\alpha_x \in \Omega^1(P)$ is defined by $\langle \alpha_x, \xi_P \rangle = -(x, \xi)$ for $x \in \mathfrak{k}$ and $\xi \in \mathfrak{p}$.

Definition 2.3. *The action of K on M is called quasi-Hamiltonian if it admits a moment map. A quasi-Hamiltonian space is a manifold with bivector on which a quasi-Poisson Lie group acts by a quasi-Hamiltonian action.*

The following lemma will be useful in this paper for the proofs of Proposition 2.8 and Theorem 2.7.

Lemma 2.4. *Let (M, w_M) be a manifold with bivector on which the compact simple Lie group K act in a quasi-Poisson manner. Then (M, w_M) is a quasi-Hamiltonian space if and only if there exists a map $\mu : M \rightarrow P$ which is equivariant with respect to action of K on M and the action of K on P by conjugation which satisfies*

$$w^\sharp(\mu^*(x, \theta)) = \frac{1}{2}((1_{\mathfrak{k}} + \text{Ad}_\mu)x)_M$$

for all $x \in \mathfrak{k}$. Here $w^\sharp : T^*M \rightarrow T_*M$ is given by $w^\sharp(\alpha) = w(\alpha, \cdot)$ for $\alpha \in T^*M$, and $\theta : T_*K \rightarrow \mathfrak{k}$ is the left-invariant Maurer-Cartan on K . For K a matrix group $\theta = k^{-1}dk$.

Proof: See [AKS, Proposition 5.33]. □

Example 2.5. *The basic example of a quasi-Hamiltonian space is the space P . The action of K on P is the dressing action and the associated moment map is the identity map. The bivector on P is given by $w_P = \frac{1}{2} \sum_i e_i^\lambda \wedge e_i^\rho$.*

In general, any K -invariant embedded submanifold of P is also a quasi-Hamiltonian space with moment map given inclusion.

Example 2.6. Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ be the standard Manin quasi-triple. Let $C \subset P$ be a conjugacy class in P . The action of K on C given by conjugation is a quasi-Poisson action. The momentum map associated to this action of is the inclusion map (i.e. $\mu : C \rightarrow P$ given by $\mu(g) = g$). Since the bivector w_P is K -invariant, the bivector on C is given by the restriction $w_P|_C$.

Even though a quasi-Hamiltonian space (M, μ, w_M) is not in general a Poisson manifold, $\frac{1}{2}[w_M, w_M] = \varphi_M$, there is still a notion of reduction to a symplectic manifold.

Lemma 2.7. Let (M, w_M, μ) be a quasi-Hamiltonian space such that the bivector w_M is everywhere nondegenerate. Assume M/G is a smooth manifold in a neighborhood U of $p(x_0)$, where $p : M \rightarrow M/G$ and $x_0 \in M$. Let $x \in M$ be such that $p(x) \in U$ and $s = \mu(x) \in D/G$ is a regular value of the moment map μ . Then the symplectic leaf through $p(x)$ in the Poisson manifold U is the connected component of the intersection with U on the projection of the manifold $\mu^{-1}(s)$.

Proof: See [AKS, Theorem 5.5.5]

2.3. Fusion product of quasi-Poisson manifolds. Given quasi-Hamiltonian spaces M_1 and M_2 each acted on by K with associated moment maps $\mu_1 : M_1 \rightarrow P$ and $\mu_2 : M_2 \rightarrow P$, it is not true that $M_1 \times M_2$ with the product bivector structure is a quasi-Hamiltonian K -space with the action being the diagonal action of K on $M_1 \times M_2$. We can define a new bivector on $M_1 \times M_2$ such that diagonal action is a quasi-Poisson action with respect to this new bivector. $M_1 \times M_2$ with this bivector is called the fusion product and is due to [AKSM].

As defined in the previous section, the subscript M denotes the vector field, or multi-vector field, induced by the action of K on M .

Proposition 2.8. Let (M_1, w_1, μ_1) and (M_2, w_2, μ_2) be quasi-Hamiltonian K -spaces in the sense of [AKS]. Then $M = M_1 \times M_2$ with the action of K on M given by the diagonal action, bivector on M given by

$$w_M = w_1 + w_2 + \frac{1}{2} \sum_j (e_j)_{M_1} \wedge (e_j)_{M_2}$$

and moment map $\mu = \mu_1 \mu_2$ is a quasi-Hamiltonian K -space. Recall $\{e_i\}$ is an orthonormal basis of \mathfrak{k} . M with this structure is called the fusion product of M_1 and M_2 and is denoted by $M = M_1 \otimes M_2$.

Proof: We begin by showing the diagonal action of K on (M, w_M) is a quasi-Poisson action. For this we need to show,

- (i) $\frac{1}{2}[w_M, w_M] = \varphi_M$
- (ii) $\mathcal{L}_{x_M} w_M = 0$.

We will then show that $\mu : M_1 \times M_2 \rightarrow P$ given above is the moment map associated to the diagonal action.

It is a straightforward calculation to show (i):

$$\begin{aligned}
\frac{1}{2}[w_M, w_M] &= \frac{1}{2}\left[w_1 + w_2 + \frac{1}{2}\sum_j (e_j)_{M_1} \wedge (e_j)_{M_2}, w_1 + w_2 + \frac{1}{2}\sum_k (e_k)_{M_1} \wedge (e_k)_{M_2}\right] \\
&= \frac{1}{2}[w_1, w_1] + \frac{1}{2}[w_2, w_2] + \left[w_1 + w_2, \frac{1}{2}\sum_{j=1}^n (e_j)_{M_1} \wedge (e_j)_{M_2}\right] \\
&\quad + \frac{1}{2}\left[\frac{1}{2}\sum_j (e_j)_{M_1} \wedge (e_j)_{M_2}, \frac{1}{2}\sum_k (e_k)_{M_1} \wedge (e_k)_{M_2}\right] \\
&= \frac{1}{2}[w_1, w_1] + \frac{1}{2}[w_2, w_2] + \left[w_1 + w_2, \sum_j (e_j)_{M_1} \wedge (e_j)_{M_2}\right] \\
&\quad + \frac{1}{8}\sum_{j,k} \left(\left[(e_j)_{M_1}, (e_k)_{M_1}\right] \wedge (e_j)_{M_2} \wedge (e_k)_{M_2} + \left[(e_j)_{M_2}, (e_k)_{M_2}\right] \wedge (e_j)_{M_1} \wedge (e_k)_{M_1}\right)
\end{aligned}$$

But $\frac{1}{2}[w_i, w_i] = \varphi_{M_i}$ for $i = 1, 2$ since the K -actions on M_1 and M_2 are quasi-Poisson actions. Also, we have $[(e_k)_{M_i}, w_i] = \mathcal{L}_{(e_k)_{M_i}} w_i = -\left(F(e_k)\right)_{M_i}$ where $F : \mathfrak{k} \rightarrow \wedge^2 \mathfrak{k}$ is the cobracket. But $F \equiv 0$ for the standard quasi-Poisson Lie group K we have, thus $[(e_k)_{M_i}, w_i] = 0$. Let f_{jk}^i denote the structure constants on \mathfrak{k} . The above equations then become

$$\begin{aligned}
&= \varphi_{M_1} + \varphi_{M_2} + 0 + \frac{1}{8}\sum_{j,k} [e_j, e_k]_{M_1} \wedge (e_j)_{M_2} \wedge (e_k)_{M_2} \\
&\quad + \frac{1}{8}\sum_{j,k} [e_j, e_k]_{M_2} \wedge (e_j)_{M_1} \wedge (e_k)_{M_1} \\
&= \frac{1}{24}\sum_{ijk} f_{jk}^i (e_i)_{M_1} \wedge (e_j)_{M_1} \wedge (e_k)_{M_1} + \frac{1}{24}\sum_{ijk} f_{jk}^i (e_i)_{M_2} \wedge (e_j)_{M_2} \wedge (e_k)_{M_2} \\
&\quad + \frac{1}{8}\sum_{ijk} f_{jk}^i (e_i)_{M_1} \wedge (e_j)_{M_2} \wedge (e_k)_{M_2} + \frac{1}{8}\sum_{ijk} f_{jk}^i (e_i)_{M_2} \wedge (e_j)_{M_1} \wedge (e_k)_{M_1} \\
&= \frac{1}{24}\sum_{ijk} f_{jk}^i \left((e_i)_{M_1} + (e_i)_{M_2}\right) \wedge \left((e_j)_{M_1} + (e_j)_{M_2}\right) \wedge \left((e_k)_{M_1} + (e_k)_{M_2}\right) \\
&= \frac{1}{24}\sum_{ijk} f_{jk}^i (e_i)_M \wedge (e_j)_M \wedge (e_k)_M \\
&= \varphi_M
\end{aligned}$$

To show (ii), we again use $\mathcal{L}_{(e_k)_{M_i}} w_{M_i} = 0$.

$$\begin{aligned}
\mathcal{L}_{(e_k)_M} w_M &= \mathcal{L}_{(e_k)_{M_1} + (e_k)_{M_2}} \left(w_1 + w_2 + \sum (e_j)_{M_1} \wedge (e_j)_{M_2} \right) \\
&= \mathcal{L}_{(e_k)_{M_1} + (e_k)_{M_2}} \left(\sum (e_j)_{M_2} \wedge (e_j)_{M_2} \right) \\
&= \sum \left[(e_k)_{M_1}, (e_j)_{M_1} \right] \wedge (e_j)_{M_2} - \sum \left[(e_k)_{M_2}, (e_j)_{M_2} \right] \wedge (e_j)_{M_1} \\
&= \sum_{i,j} C_{kj}^i (e_i)_{M_1} \wedge (e_j)_{M_2} - \sum_{i,j} C_{kj}^i (e_i)_{M_2} \wedge (e_j)_{M_1} \\
&= 0
\end{aligned}$$

We next use Lemma 2.4 to show that $\mu = \mu_1 \mu_2 : M_1 \times M_2 \rightarrow P$ is indeed the moment map associated to the diagonal action.

$$\begin{aligned}
w^\sharp(\mu^*(x, \theta)) &= w^\sharp((\mu_1 \mu_2)^*(x, \theta)) \\
&= w^\sharp((x, \mu_2^* \theta + Ad_{\mu_2^{-1}} \mu_1^* \theta)) \\
&= w^\sharp(\mu_2^*(x, \theta) + \mu_1^*(Ad_{\mu_2} x, \theta)) \\
&= w_1^\sharp(\mu_1^*(Ad_{\mu_2} x, \theta)) + w_2^\sharp(\mu_2^*(x, \theta)) + \frac{1}{2} \sum_j \left((\mu_1^*(Ad_{\mu_2} x, \theta))(e_j)_{M_1} \right) (e_j)_{M_2} \\
&\quad - \frac{1}{2} \sum_j \left((\mu_2^*(x, \theta))(e_j)_{M_2} \right) (e_j)_{M_1}
\end{aligned}$$

(M_i, w_i) is a quasi-Hamiltonian space with moment map $\mu_i : M_i \rightarrow P_i$, so we have by Lemma 2.4

$$w_i^\sharp(\mu_i^*(x, \theta)) = \frac{1}{2}((1 + Ad_{\mu_i})x)_{M_i}.$$

We can also see that

$$\begin{aligned}
\sum_i \left((\mu_j^*(x, \theta))(e_i)_{M_j} \right) (e_i)_{M_k} &= \sum_i (x, Ad_{\mu_j^{-1}} e_i - e_i)(e_i)_{M_k} \\
&= \sum_i (Ad_{\mu_j} x - x, e_i)(e_i)_{M_k} \\
&= (Ad_{\mu_j} x - x)_{M_k}
\end{aligned}$$

So the above becomes

$$\begin{aligned}
w^\sharp(\mu^*(X, \theta)) &= \frac{1}{2}(Ad_{\mu_2} + Ad_{\mu_1\mu_2}X)_{M_1} + \frac{1}{2}(1 + Ad_{\mu_2}X)_{M_2} + \frac{1}{2}(Ad_{\mu_1\mu_2}X - Ad_{\mu_2}X)_{M_2} \\
&\quad - \frac{1}{2}(Ad_{\mu_2}X - X)_{M_1} \\
&= \frac{1}{2}((1 + Ad_{\mu_1\mu_2})X)_{M_1} + \frac{1}{2}((1 + Ad_{\mu_1\mu_2})X)_{M_2} \\
&= \frac{1}{2}((1 + Ad_{\mu_1\mu_2})X)_M
\end{aligned}$$

□

Remark 2.9. *It is a quick calculation to show the fusion product is associative, that is $M_1 \otimes (M_2 \otimes M_3) \simeq (M_1 \otimes M_2) \otimes M_3$. The bivector is given by*

$$w = w_1 + w_2 + w_3 + \frac{1}{2} \sum_i (e_i)_{M_1} \wedge (e_i)_{M_2} + \frac{1}{2} \sum_i (e_i)_{M_1} \wedge (e_i)_{M_3} + \frac{1}{2} \sum_i (e_i)_{M_2} \wedge (e_i)_{M_3}.$$

The quasi-Hamiltonian space we are most interested in for this paper is the fusion product of n conjugacy classes in P . Recall from Example 2.6 that $C_{r_i} \subset P$ is a quasi-Hamiltonian space with action given by conjugation and the associated moment map given by inclusion. The fusion product of n conjugacy classes $C_r^n = C_{r_1} \otimes \cdots \otimes C_{r_n}$, $r = (r_1, \dots, r_n) \in \mathbb{R}_+$ is also a quasi-Hamiltonian space with action given by the diagonal conjugation and moment map $\tilde{\mu} : M \rightarrow P$ given by multiplication, $\tilde{\mu}(g_1, g_2, \dots, g_n) = g_1 g_2 \cdots g_n$. The bivector on this space is given by

$$\tilde{w} = \frac{1}{2} \sum_{i=1}^n \sum_k \left(e_k^\lambda \wedge e_k^\rho \right)_i + \frac{1}{2} \sum_{i < j}^n \sum_k \left(e_k^\lambda - e_k^\rho \right)_i \wedge \left(e_k^\lambda - e_k^\rho \right)_j$$

where the subscripts i, j denote the vector field on $C_{r_i}, C_{r_j} \subset C_r^n$.

2.4. Poisson bracket on $C^\infty(P^n)^K$. For a general quasi-Hamiltonian space (M, w_M) , the bracket on $C^\infty(M)$ defined by the bivector w_M is not a Poisson bracket. This is easy to see since the Shouten bracket $[w_M, w_M] = \varphi_M$ is an invariant trivector field. The bracket does however define a Poisson bracket when we restrict to the space $C^\infty(M)^K$ of smooth K -invariant functions on M .

Lemma 2.10. *Let K be a connected quasi-Poisson Lie group acting on a manifold (M, w_M) in a quasi-Poisson manner. Then the bivector w_M defines a Poisson bracket on the space $C^\infty(M)^K$ of the smooth K -invariant functions in M .*

Proof: See [AKS, Theorem 4.2.2]

□

For $\psi \in C^\infty(P^n)$ we define

$$D_i \psi : P^n \rightarrow \mathfrak{k}_i, \quad D'_i \psi : P^n \rightarrow \mathfrak{k}_i$$

as follows. Let $g = (g_1, \dots, g_n) \in P^n$ and $x = (x_1, \dots, x_n) \in \mathfrak{k}^n$, then

$$d_i \psi_g(x^\rho) = (D_i \psi, x) = \frac{d}{dt} \Big|_{t=0} \psi(g_1, \dots, e^{tx_i} g_i, \dots, g_n)$$

$$d_i \psi_g(x^\lambda) = (D'_i \psi, x) = \frac{d}{dt} \Big|_{t=0} \psi(g_1, \dots, g_i e^{tx_i}, \dots, g_n).$$

Here $(,)$ is the Killing form extended to \mathfrak{k}^n by $(x, y) = \sum_{i=1}^n (x_i, y_i)$ for $x, y \in \mathfrak{k}^n$.

Remark 2.11. *It is easy to see that*

$$Ad_{g_i} D'_i \psi(g) = D_i \psi$$

We also define

$$\Psi_j(g) = \sum_{i=1}^{j-1} [D_i \psi(g) - D'_i \psi(g)] + D_j \psi(g)$$

We now define the Poisson bracket on $C^\infty(P^n)^K$.

Proposition 2.12. *Let $\phi, \psi \in C^\infty(P^n)^K$ then*

$$\{\phi, \psi\}(g) = \sum_{j=1}^n (D'_j \phi(g) - D_j \phi(g), \Psi_j(g))$$

Proof:

Let us first note that for $x, y \in \mathfrak{k}$ $\sum_i (x, e_i)(y, e_i) = (x, y)$. Now,

$$\begin{aligned} \{\varphi, \psi\}(g) &= w(d\varphi, d\psi) \\ &= \frac{1}{2} \sum_{i=1}^n \sum_k (e_k^\lambda \wedge e_k^\rho)_i (d\phi, d\psi) + \frac{1}{2} \sum_{i < j} \sum_k ((e_k^\lambda - e_k^\rho)_i \wedge (e_k^\lambda - e_k^\rho)_j) (d\phi, d\psi) \\ &= \frac{1}{2} \sum_{i=1}^n \sum_k d_i \phi(e_k^\lambda) d_i \psi(e_k^\rho) - d_i \phi(e_k^\rho) d_i \psi(e_k^\lambda) \\ &\quad + \frac{1}{2} \sum_{i < j} \sum_k d_i \phi(e_k^\lambda - e_k^\rho) d_j \psi(e_k^\lambda - e_k^\rho) - d_j \phi(e_k^\lambda - e_k^\rho) d_i \psi(e_k^\lambda - e_k^\rho) \\ &= \frac{1}{2} \sum_{i=1}^n \sum_k (D'_i \phi, e_k) (D_i \psi, e_k) - (D_i \phi, e_k) (D'_i \psi, e_k) \\ &\quad + \frac{1}{2} \sum_{i < j} \sum_k (D'_i \phi - D_i \phi, e_k) (D'_j \psi - D_j \psi, e_k) - (D'_j \phi - D_j \phi, e_k) (D'_i \psi - D_i \psi, e_k) \\ &= \frac{1}{2} \sum_{i=1}^n (D'_i \phi, D_i \psi) - (D_i \phi, D'_i \psi) \\ &\quad + \frac{1}{2} \sum_{i < j} (D'_i \phi - D_i \phi, D'_j \psi - D_j \psi) - (D'_j \phi - D_j \phi, D'_i \psi - D_i \psi) \\ &= \frac{1}{2} \sum_{i=1}^n (D'_i \phi, D_i \psi) - (D_i \phi, D'_i \psi) \\ &\quad + \frac{1}{2} \sum_{i < j} (D'_i \phi - D_i \phi, D'_j \psi - D_j \psi) - \sum_{i > j} (D'_i \phi - D_i \phi, D'_j \psi - D_j \psi) \end{aligned}$$

But since $\psi \in C^\infty(P^n)^K$ is K -invariant, a quick calculation shows

$$\sum_{i=1}^n [D_i \psi - D'_i \psi] = 0$$

Using this fact and also that $(D'_i \phi, D'_i \psi) = (D_i \phi, D_i \psi)$ for all i , we can rewrite the above as,

$$\begin{aligned} \{\phi, \psi\} &= \frac{1}{2} \sum_{i=1}^n \left(D'_i, \phi - D_i \phi, D_i \psi + D'_i \psi \right) \\ &\quad - \frac{1}{2} \sum_{i \geq j} \left(D'_i \phi - D_i \phi, D'_j \psi - D_j \psi \right) - \frac{1}{2} \sum_{i > j} \left(D'_i \phi - D_i \phi, D'_j \psi - D_j \psi \right) \\ &= \sum_{i=1}^n \left(D'_i \phi - D_i \phi, \Psi_i \right) \end{aligned}$$

□

From the above Proposition we can also define the Hamiltonian vector field X_ψ associated to $\psi \in C^\infty(P^n)^K$ by $X_\psi = w^\sharp(d\psi)$.

Corollary 2.13. *The Hamiltonian vector field $X_\psi(g) = ((X_1(g), \dots, X_n(g)))$ associated to the K -invariant function $\psi \in C^\infty(P^n)^K$ is given by*

$$X_j(g) = dL_{g_j} \Psi_j - dR_{g_j} \Psi_j, \quad 1 \leq j \leq n.$$

and $g = (g_1, g_2, \dots, g_n)$.

Proof: We use the convention $\{\phi, \psi\} = d\phi(X_\psi) = \sum_{j=1}^n d_j \phi((X_j(g)))$. Proposition 2.12 gives us

$$\begin{aligned} d\phi(X_\psi(g)) &= \{\phi, \psi\} \\ &= \sum_{j=1}^n \left(D'_j \phi - D_j \phi, \Psi_j \right) \\ &= \sum_{j=1}^n d_j \phi(dL_{g_j} \Psi_j) - d_j \phi(dR_{g_j} \Psi_j) \\ &= \sum_{j=1}^n d_j \phi(dL_{g_j} \Psi_j - dR_{g_j} \Psi_j) \end{aligned}$$

□

3. THE SYMPLECTIC STRUCTURE ON $M_r(\mathbb{S}^3)$

Throughout the rest of the paper, we let $G = SU(2) \times SU(2)$, $K = SU(2)$, and $P \simeq SU(2)$. In this section, we will define a symplectic structure on M_r obtained from the reduction of the fusion product of conjugacy classes to a symplectic manifold.

Recall, we defined $Pol_n(*)$ to be the open n -gons in \mathbb{S}^3 with side-length less than π , so that we can choose an unique geodesic between vertices. The map $\Phi : P^n \rightarrow Pol_n(*) \subset (S^3)^n$ defined by

$$\Phi(g) = (*, g_1*, g_1g_2*, \dots, g_1g_2 \cdots g_n*)$$

is a diffeomorphism.

Proposition 3.1. *The map Φ is a K -equivariant diffeomorphism where K acts on P^n by the dressing action (diagonal conjugation) and on $Pol_n(*)$ by the diagonal action on $(\mathbb{S}^3)^n$.*

Proof: $*$ $\in P$ is an element in P which is fixed by the K -action, that is $Ad_k(*) = *$ for all $k \in K$. For $k \in K$ and $g \in P^n$, $k \cdot p = (Ad_k g_1, \dots, Ad_k g_n)$, so

$$\begin{aligned} \Phi(k \cdot g) &= (*, Ad_k(g_1)*, \dots, Ad_k(g_1 \cdots g_n)*) \\ &= (Ad_k*, Ad_k(g_1*), \dots, Ad_k(g_1 \cdots g_n*)) \\ &= k \cdot (*, g_1*, \dots, g_1 \cdots g_n*). \end{aligned}$$

□

Remark 3.2. *The map Φ induces a diffeomorphism from $\{g \in P^n : g_1 \cdots g_n = 1\}$ to $CPol(*)$.*

We have seen that the K -orbits in a quasi-Hamiltonian space are quasi-Hamiltonian spaces. In particular, a conjugacy class $C \subset P$ is a quasi-Hamiltonian space. Let $r \in \mathbb{R}^n$, with $r = (r_1, \dots, r_n)$. Let $C_{r_i} \subset P$ denote the conjugacy class in P such that $r_i = d(*, g_i*) = \cos^{-1} \left(-\frac{1}{2} \text{trace}(g_i) \right) \in \mathbb{R}$ for all $g_i \in C_{r_i}$.

Lemma 3.3. *The map Φ induces a K -equivariant diffeomorphism from $C_{r_1} \times \cdots \times C_{r_n}$ to \tilde{N}_r , the space of open n -gons with fixed side-lengths based at $*$, where $r_i = d(g_1 \cdots g_i*, g_1 \cdots g_{i-1}*)$, for all $1 \leq i \leq n$.*

Proof: Follows from the fact that k fixes side-lengths. □

Corollary 3.4. *Φ induces a diffeomorphism from the space $\{g \in C_r^n : g_1 \cdots g_n = 1\}/K$ to M_r the moduli space of closed n -gons in \mathbb{S}^3 .*

In §2.3 we saw that the fusion product of n conjugacy classes in P , $(C_r^n, \tilde{\mu}, \tilde{w})$, is a quasi-Hamiltonian space with the moment map $\tilde{\mu}$ given by multiplication. So, $\tilde{\mu}^{-1}(1)/K = \{g \in C_r^n : g_1 \cdots g_n = 1\}/K$. We must determine when this restriction and quotient gives rise to symplectic manifold. Lemma 2.7 tells us that $\tilde{\mu}^{-1}(1)/K$ is a symplectic manifold when

- \tilde{w} is everywhere nondegenerate on C_r^n
- 1 is a regular value of $\tilde{\mu}$.

We use the following remark from [AKS, Example 5.5.4] to give the nondegeneracy condition.

Remark 3.5. *Let K be a quasi-Poisson Lie group arising from the standard quasi-triple and (M, μ, w) is a quasi-Hamiltonian space. Then (M, μ, w) is nondegenerate if and only if, for each $m \in M$,*

$$\ker(w_m^\sharp) = \{\mu^*(x, \theta) : x \in \ker(1 + Ad_{\mu(m)})\}.$$

Here $x \in \mathfrak{k}$.

It follows that the fusion product of conjugacy classes is nondegenerate.

Lemma 3.6. *1 is a regular value of $\tilde{\mu}$ if and only if $\mathfrak{k}_g = \{x \in \mathfrak{k} : x_{C_r^n} = 0\} = 0$ for all $g \in \tilde{\mu}^{-1}(1)$.*

Proof: We refer to Lemma 2.4. Let $x \in \mathfrak{k}$. Then $x \in (Im(d\tilde{\mu}|_g))^\perp \Leftrightarrow (x, \tilde{\mu}^*\theta) = 0 \Leftrightarrow 0 = \tilde{w}^\sharp((x, \tilde{\mu}^*\theta)) = ((1 + Ad_{\tilde{\mu}(g)})x)_{C_r^n} = (2x)_{C_r^n}$. \square

A polygon is said to be degenerate if it can be contained in a geodesic in \mathbb{S}^3 . It follows from the above lemma that if there does not exist $g \in \tilde{\mu}^{-1}(1) \subset C_r^n$ such that $\Phi(g)$ is a degenerate polygon, then 1 is a regular value of $\tilde{\mu}$.

Theorem 3.7. *The moduli space M_r containing no degenerate polygons has a symplectic structure which is the transport structure from the moduli space $\mu^{-1}(1)/K$.*

In §6, we need a formula for the symplectic form on M_r in §6.

Remark 3.8. *The symplectic form is given by*

$$\tilde{\omega} = \sum_{i=1}^n \omega_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=i+1}^n (Ad_{g_1 \cdots g_{i-1}} \bar{\theta}_i \wedge_b Ad_{g_1 \cdots g_{j-1}} \bar{\theta}_j).$$

where ω_i is the quasi-Hamiltonian 2-form on the conjugacy class $C_i \subset SU(2)$, see [AMM1], and $\bar{\theta}_i$ is the right-invariant Maurer-Cartan form on $C_i \subset SU(2)$. We denote by \wedge_b the wedge product together with the killing form on G .

4. BENDING HAMILTONIANS

4.1. Hamiltonian vector fields. Recall, $K = SU(2)$ and $C_r^n = C_{r_1} \otimes \cdots \otimes C_{r_n}$, where $C_{r_i} \subset P$ is a conjugacy class in $P \simeq SU(2)$. Let $(x, y) = -\frac{1}{2}Tr(xy)$. In this section we will compute the Hamiltonian vector fields X_{f_j} associated to the functions $f_i \in C^\infty(C_r^n)^K$ given by

$$f_j(g) = tr(g_1 \cdots g_j), \quad 1 \leq j \leq n.$$

See §2.4 for the definition of the Poisson bracket on $C^\infty(C_r^n)^K$. We leave it to the reader to verify the following lemma.

Lemma 4.1.

$$\begin{aligned} D_{i+1}f_j(g) &= D'_i f_j(g), \quad 1 \leq i \leq j-1 \\ D_1f_j(g) &= D'_j f_j(g) \end{aligned}$$

for all $1 \leq j \leq n$.

We define $F_j : P \rightarrow \mathfrak{k}$ by

$$F_j(g) = \left((g_1 \cdots g_j) - (g_1 \cdots g_j)^{-1} \right).$$

We then have the following lemma.

Lemma 4.2. $F_j(g) = D_1f_j(g)$

Proof: For $g \in C_r^n$ and $X \in \mathfrak{k}$

$$\begin{aligned} (D_1 f_j(g), X) &= \left. \frac{d}{dt} \right|_{t=0} \text{tr}(e^{tX} g_1 g_2 \cdots g_j) \\ &= \text{tr}(X g_1 g_2 \cdots g_j) \\ &= \text{tr}(g_1 g_2 \cdots g_j X) \end{aligned}$$

but since

$$\text{tr}((g_1 g_2 \cdots g_j)^{-1} X) = \text{tr}((g_1 \cdots g_j)^* X) = \text{tr}(X^* g_1 \cdots g_j) = -\text{tr}(g_1 \cdots g_j X)$$

it follows that

$$\begin{aligned} \text{tr}(g_1 g_2 \cdots g_j X) &= \frac{1}{2} \text{tr} \left(((g_1 g_2 \cdots g_j) - (g_1 \cdots g_j)^{-1}) X \right) \\ &= \left(-((g_1 \cdots g_j) - (g_1 \cdots g_j)^{-1}), X \right). \end{aligned}$$

Since $-((g_1 \cdots g_j) - (g_1 \cdots g_j)^{-1}) \in \mathfrak{k}$ and (\cdot, \cdot) is a nondegenerate bilinear form, we have

$$D_1 f_j(g) = -((g_1 \cdots g_j) - (g_1 \cdots g_j)^{-1}) = -F_j(g). \quad \square$$

We have the following formula of the Hamiltonian vector fields X_{f_i} .

Theorem 4.3. *The Hamiltonian vector field X_{f_i} has an i -th component given by*

$$(X_{f_j}(g))_i = dR_{g_i} F_j(g) - dL_{g_i} F_j(g), \quad 1 \leq i \leq j,$$

$$(X_{f_j}(g))_i = 0, \quad j < i \leq n$$

Proof: Recall from Corollary 2.13 that for $\psi \in C^\infty(C_r^n)^K$, $X_\psi(g)$ is given by

$$(X_\psi(g))_i = dL_{g_i} \Psi_i(g) - dR_{g_i} \Psi_i(g)$$

where $\Psi_i(g) = D_1 \psi(g) - D'_1 \psi(g) + D_2 \psi(g) - \cdots - D_{i-1} \psi(g) + D_i \psi(g)$. This together with Lemma 4.1 gives us

$$(X_{f_j}(g))_i = dL_{g_i} D_1 f_j(g) - dR_{g_i} D_1 f_j(g), \quad 1 \leq i \leq j$$

and

$$(X_{f_j}(g))_i = 0, \quad j < i \leq n.$$

But from Lemma 4.2, $-F_j(g) = D_1 f_j(g)$, completing the proof. \square

4.2. Commuting flows. In this section we will show the family of Hamiltonians $\{f_j\}_{j=1}^n$ Poisson commute for $1 \leq j \leq n$.

Proposition 4.4. $\{f_i, f_j\} \equiv 0$ for all i, j .

Proof: Without loss of generality we may assume $i < j$, then by Proposition 2.12

$$\begin{aligned}
\{f_i, f_j\}(g) &= \sum_{k=1}^j \left(D'_k f_i(g) - D_k f_i(g), F_j(g) \right) \\
&= - \left(\sum_{k=1}^j (D'_k f_i(g) - D_k f_i(g)), F_j(g) \right) \\
&= \left(0, F_j(g) \right) \\
&= 0
\end{aligned}$$

Here we used $\sum_{k=1}^i (D_k f_i - D'_k f_i) = 0$. □

4.3. Hamiltonian flow. In this section we will calculate the Hamiltonian flow, Φ_j^t , associated to f_j . Recall that the Hamiltonian flow is the solution to the ODE

$$(*) \begin{cases} \frac{dg_i}{dt} = dR_{g_i} F_j(g) - dL_{g_i} F_j(g), & 1 \leq i \leq j \\ \frac{dg_i}{dt} = 0, & j < i \leq n \end{cases}$$

Lemma 4.5. $F_j(g)$ is invariant along solution curves of $(*)$.

Proof: To prove the lemma, it suffices to show that $\psi_j(g) = g_1 \cdots g_j$ is invariant along solution curves of $(*)$.

$$\begin{aligned}
\frac{d}{dt} \psi_j(g(t)) &= \frac{d}{dt} (g_1(t) g_2(t) \cdots g_j(t)) \\
&= \frac{dg_1}{dt}(t) g_2(t) \cdots g_j(t) + g_1(t) \frac{dg_2}{dt}(t) \cdots g_j(t) + \cdots + g_1(t) g_2(t) \cdots \frac{dg_j}{dt}(t) \\
&= [F_j(g(t)) g_1(t) - g_1(t) F_j(g(t))] g_2(t) \cdots g_j(t) + g_1(t) [F_j(g(t)) g_2(t) - g_2(t) F_j(g(t))] \cdots g_j(t) \\
&= g_1(t) g_2(t) \cdots [F_j(g(t)) g_j(t) - g_j(t) F_j(g(t))] \\
&= F_j(g(t)) g_1(t) \cdots g_j(t) - g_1(t) \cdots g_j(t) F_j(g(t)) \\
&= 0
\end{aligned}$$

□

Lemma 4.6. The curve $\exp(t F_j(g))$ is periodic with period $2\pi/\sqrt{4 - f_j^2}$.

Proof: Left to reader. □

We are now able to find the Hamiltonian flow Φ_j^t .

Theorem 4.7. The Hamiltonian flow, Φ_j^t , associated to the Hamiltonian f_j given by $\Phi_j^t(g) = (\tilde{g}_1(t), \dots, \tilde{g}_n(t))$ where

$$\tilde{g}_i(t) = \begin{cases} \text{Ad}(\exp(t F_j(g))) g_i, & 1 \leq i \leq j \\ g_i, & j < i \leq n. \end{cases}$$

The flow is periodic with period $2\pi/\sqrt{4 - f_j^2}$.

The flows $\{\Phi_j^t\}$ do not give rise to a torus action on M_r since they do not have constant period. We now look at the length functions $\ell_j(g) = \cos^{-1}(-\frac{1}{2}f_j(g))$. Then

$$d\ell_j = \frac{1}{\sqrt{4 - f_j^2}} df_j$$

and

$$X_{\ell_j} = \frac{1}{\sqrt{4 - f_j^2}} X_{f_j}.$$

It is not difficult to see that the family of functions $\{\ell_j\}_{j=2}^{n-1}$ also Poisson commute, but their Hamiltonian flows are not everywhere defined. If we restrict to the space M'_r such $\ell_j \neq 0$ or $\ell_j \neq \pi$ for all diagonals in M_r . The Hamiltonian flows $\{\Psi_j^t\}$ on M'_r associated to $\{\ell_j\}$ are periodic with constant period 2π and constant angular velocity 1. These flows define a Hamiltonian $(n-3)$ -torus action on the space M'_r .

5. BRAID ACTION ON M_r

There exists an action of the pure braid group P_n on the manifold M_r which preserves the symplectic structure. In this section, we show that the generators of the pure braid group arise as the time 1 Hamiltonian flows of the family of functions $h_{ij}, 1 \leq i < j \leq n-1$ where $h_{ij} \in C^\infty(M_r)^K$ is defined by,

$$h_{ij}(g) = \frac{1}{2} \left(\cos^{-1} \left(-\frac{1}{2} \text{tr}(g_i g_j) \right) \right)^2.$$

Let C_{12} denote $C_1 \otimes C_2$, where $C_i \subset P$ is a conjugacy class. Let w_{12} denote the quasi-Poisson bivector on C_{12} . We have the following proposition.

Proposition 5.1. *The diffeomorphism $R : C_1 \otimes C_2 \rightarrow C_2 \otimes C_1$ given by $R(g_1, g_2) = (Ad_{g_1} g_2, g_1)$ is a bivector map taking w_{12} to w_{21} .*

Remark 5.2. *The diffeomorphism $R' : C_1 \otimes C_2 \rightarrow C_2 \otimes C_1$ given by $R'(g_1, g_2) = (g_2, Ad_{g_2^{-1}} g_1)$ is also a bivector map taking w_{12} to w_{21} .*

Remark 5.3. $R \circ R' = Id_{C_1 \otimes C_2} = R' \circ R$

We now define $R_i : C_1 \otimes \cdots \otimes (C_i \otimes C_{i+1}) \otimes \cdots \otimes C_n \rightarrow C_1 \otimes \cdots \otimes (C_{i+1} \otimes C_i) \otimes \cdots \otimes C_n$ to be the map given by

$$R_i(g_1, \dots, g_i, g_{i+1}, \dots, g_n) = (g_1, \dots, Ad_{g_i} g_{i+1}, g_i, \dots, g_n)$$

that is, R applied to the i th and $(i+1)$ th term of M_r . R'_i can be defined in a similar way.

Lemma 5.4. *The full braid group B_n has a faithful representation as a group of automorphism of the closed n -gons in S^3 in which side-lengths are fixed but the order of the sides is not fixed. The generators of B_n are given by $R_i, 1 \leq i \leq n-1$.*

We now restrict B_n to P_n to get an action of the pure braid group on C_r^n . This action induces a symplectomorphism on the moduli space M_r .

Corollary 5.5. *Let $A_{ij} = R_{j-1} \circ \cdots \circ R_{i+1} \circ R_i^2 \circ R'_{i+1} \circ \cdots \circ R'_{j-1}$, $1 \leq i < j \leq n$. A_{ij} induces a symplectomorphism from M_r to itself. A_{ij} , $1 \leq i < j \leq n$ are the generators of P_n which has a faithful representation as a group of automorphisms of M_r .*

We will now show that the braid group actions A_{ij} can be realized as the time one Hamiltonian flows of the Hamiltonians h_{ij} given at the start of the section. We begin by studying the Hamiltonian flows associated to the functions $f_{ij} \in C^\infty(C_r^n)^K$ given by $f_{ij}(g) = \text{tr}(g_i g_j)$. Define $F_{ij} : C_r^n \rightarrow \mathfrak{k}$ by $F_{ij}(g) = ((g_i g_j) - (g_i g_j)^{-1})$.

The Hamiltonian flow associated to f_{ij} is given by $\Phi_{ij}^t(g) = (\hat{g}_1(t), \dots, \hat{g}_n(t))$ where

$$\hat{g}_k(t) = \begin{cases} g_k, & 0 < k < i \text{ and } j < k < n+1 \\ \text{Ad}\left(\exp(tF_{ij}(g))\right)g_k, & k = i, j \\ \text{Ad}\left(\exp(tF_{ij}(g))g_j \exp(-tF_{ij}(g))g_j^{-1}\right)g_k, & i < k < j. \end{cases}$$

The following formula is used to relate Φ_{ij}^t to A_{ij} .

Lemma 5.6.

$$\exp\left(\frac{\cos^{-1}(-\frac{1}{2}\text{tr}(g))}{\sqrt{4 - \text{tr}^2(g)}}(g - g^{-1})\right) = g$$

We now notice that for time $t = \frac{\cos^{-1}(-\frac{1}{2}f_{ij}(g))}{\sqrt{4 - f_{ij}^2(g)}}$,

$$\Phi_{ij}^t = A_{ij}.$$

The time for which the Φ_{ij}^t flows depends on the point in M_r at which flow begins. We would like time to be independent on the starting point. We can achieve this by taking the Hamiltonian flows of the functions $h_{ij} = \frac{1}{2}(\cos^{-1}(-\frac{1}{2}f_{ij}))^2$. The Hamiltonian flow $\tilde{\Phi}_{ij}^t$ associated to h_{ij} is the renormalization of the flow Φ_{ij}^t so that

$$\tilde{\Phi}_{ij}^1 = A_{ij}$$

on M_r . We can see the pure braid group as the integer points in the Hamiltonian flows $\tilde{\Phi}_{ij}^t$, $1 \leq i < j \leq n$.

6. CONNECTION WITH SYMPLECTIC FORMS ON RELATIVE CHARACTER VARIETIES OF n -PUNCTURED 2-SPHERES

In this section, we relate the symplectic form on $M_r(\mathbb{S}^3)$ given in Remark 3.8 to the symplectic form of Goldman type obtained from the description of $M_r(\mathbb{S}^3)$ as the moduli space of flat connections on an n -punctured 2-sphere. We follow the arguments of Kapovich and Millson [KM1, §5] which considers the analogous question for $M_r(\mathbb{E}^3)$. We begin with the general case in which G is any Lie group with Lie algebra \mathfrak{g} which admits a nondegenerate, G -invariant, symmetric, bilinear form.

6.1. Relative characteristic varieties and parabolic cohomology. Let $\Sigma = \mathbb{S}^2 - \{p_1, \dots, p_n\}$ denote the n -punctured 2-sphere and U_1, \dots, U_n be disjoint disc neighborhoods of p_1, \dots, p_n , respectively. Further, Γ is the fundamental group of Σ with generators γ_i , $T = \{\Gamma_1, \dots, \Gamma_n\}$ is the collection of subgroups of Γ with Γ_i the cyclic subgroup generated by γ_i , and $U = U_1 \cup \cdots \cup U_n$.

Fix $\rho_0 \in \text{Hom}(\Gamma, G)$ a representation. In [KM2] the relative representation variety $\text{Hom}(\Gamma, T; G)$ is defined as the representations $\rho : \Gamma \rightarrow G$ such that $\rho|_{\Gamma_i}$ is contained in the closure of the conjugacy class of $\rho_0|_{\Gamma_i}$.

Remark 6.1. *If $G = SU(2)$, there exists a ρ_0 such that the relative character variety $\text{Hom}(\Gamma, T; G)/G$ is isomorphic to $M_r(\mathbb{S}^3)$. We will make this isomorphism explicit later on.*

Let $\rho \in \text{Hom}(\Gamma, T; G)$. Then ρ induces a flat principal G -bundle over Σ . The associated flat Lie algebra bundle will be denoted by $ad P$.

We define the parabolic cohomology, $H_{par}^1(\Sigma, ad P)$ to be the subspace of the de Rham cohomology classes in $H_{DR}^1(\Sigma, ad P)$ whose restrictions to each U_i are trivial.

6.2. Gauge theoretic description of the symplectic form. Let b be the nondegenerate, G -invariant, symmetric, bilinear form on \mathfrak{g} . A skew symmetric bilinear form

$$B : H_{par}^1(\Sigma, ad P) \times H_{par}^1(\Sigma, ad P) \rightarrow H^2(\Sigma, U; \mathbb{R})$$

is defined by taking the wedge product together with the bilinear form b . Evaluating on the relative fundamental class of Σ gives the skew symmetric form,

$$A : H_{par}^1(\Sigma, ad P) \times H_{par}^1(\Sigma, ad P) \rightarrow \mathbb{R}.$$

Poincare duality give us nondegeneracy of A , so A is a symplectic form on $\text{Hom}(\Gamma, T; G)$. We will show A corresponds to the symplectic form $\tilde{\omega}$ given in Remark 3.8.

We first pass through the group cohomology description of $H_{par}^1(\Sigma, ad P)$ to make this correspondence explicit.

We identify the universal cover of Σ , denoted $\tilde{\Sigma}$, with the hyperbolic plane, \mathbb{H}^2 . Let $p : \tilde{\Sigma} \rightarrow \Sigma$ by the covering projection. We define the $\mathcal{A}^\bullet(\tilde{\Sigma}, p^*Ad P)$ with $\mathcal{A}^\bullet(\tilde{\Sigma}, \mathfrak{g})$ by parallel translation from a point x_0 . Given $[\eta] \in H^1(\Sigma, ad P)$ choose a representing closed 1-form $\eta \in \mathcal{A}^1(\Sigma, ad P)$. Let $\tilde{\eta} = p^*\eta$. Then there is a unique function $f : \tilde{\Sigma} \rightarrow \mathfrak{g}$ satisfying:

- $f(x_0) = 0$
- $df = \tilde{\eta}$

A 1-cochain $h(\eta) \in C^1(\Gamma, \mathfrak{g})$ is defined by

$$h(\eta)(\gamma) = f(x) - Ad_\rho(\gamma)f(\gamma^{-1}x).$$

This induces an isomorphism from $H^1(\Sigma, ad P)$ to $H^1(\Gamma, \mathfrak{g})$. It can be seen that $[\eta] \in H_{par}^1(\Sigma, ad P)$ if and only if $h(\eta)$ restricted to Γ_i is exact for all i . That is, there exists an $x_i \in \mathfrak{g}$ such that $h(\eta)(\gamma_i^k) = x_i - Ad_\rho(\gamma_i^k)x_i$ for each γ_i a generator of Γ .

We construct the fundamental domain \mathcal{D} for Γ operating on \mathbb{H}^2 as in [KM1]. Choose x_0 on Σ and make cuts along geodesics from x_0 to the cusps. The resulting fundamental domain \mathcal{D} is a geodesic $2n$ -gon with vertices v_1, \dots, v_n and cusps $v_1^\infty, \dots, v_n^\infty$ ordered so that as we proceed clockwise around $\partial\mathcal{D}$ we see $v_1, v_1^\infty, \dots, v_n, v_n^\infty$. The generator γ_i fixes v_i^∞ and satisfies $\gamma_i v_{i+1} = v_i$. Let e_i be the oriented edge joining v_i to v_i^∞ and \hat{e}_i be the oriented edge joining v_i^∞ to v_{i+1} . Then $\gamma_i \hat{e}_i = -e_i$.

Let $\rho \in \text{Hom}(\Gamma, T; G)$ and $c, c' \in T_\rho(\text{Hom}(\Gamma, T; G)/G) \simeq H_{par}^1(\Gamma, \mathfrak{g})$ be tangent vectors at ρ . The corresponding elements in $H_{par}^1(\Sigma, ad P)$ are denoted α and α' . So $f : \Sigma \rightarrow \mathfrak{g}$

which satisfies $df = \tilde{\alpha}$ and $f_i(x_0) = 0$. Let $f(v_i^\infty) = x_i$. Then

$$\begin{aligned} c(\gamma_i) &= f(x) - \text{Ad}_{\rho(\gamma_i)} f(\gamma_i^{-1}x) \\ &= f(v_i^\infty) - \text{Ad}_{\rho(\gamma_i)} f(\gamma_i^{-1}v_i^\infty) \\ &= f(v_i^\infty) - \text{Ad}_{\rho(\gamma_i)} f(v_i^\infty) \\ &= x_i - \text{Ad}_{\rho(\gamma_i)} x_i. \end{aligned}$$

There is an equivalent formulas for c' , α' , and f' with $f'(v_i^\infty) = x'_i$.

Let $B_\bullet(\Gamma)$ be the bar resolution of Γ . Thus $B_k(\Gamma)$ is the free $\mathbb{Z}[\Gamma]$ -module on the symbols $[\gamma_1|\gamma_2|\cdots|\gamma_k]$ with

$$\partial[\gamma_1|\gamma_2|\cdots|\gamma_k] = \gamma_1[\gamma_2|\cdots|\gamma_k] + \sum_{i=1}^{k-1} (-1)^i [\gamma_1|\cdots|\gamma_i\gamma_{i+1}|\cdots|\gamma_k] + (-1)^k [\gamma_1|\cdots|\gamma_{k-1}].$$

Let $C_k(\Gamma) = B_k(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}$ with $\mathbb{Z}[\Gamma]$ acting on \mathbb{Z} by the homomorphism ϵ defined by

$$\epsilon\left(\sum_{i=1}^m a_i \gamma_i\right) = \sum_{i=1}^m a_i.$$

Then $C_k(\gamma)$ is the free abelian group on the symbols $(\gamma_1|\cdots|\gamma_k) = [\gamma_1|\gamma_2|\cdots|\gamma_k] \otimes 1$ with

$$\partial(\gamma_1|\gamma_2|\cdots|\gamma_k) = (\gamma_2|\cdots|\gamma_k) + \sum_{i=1}^{k-1} (-1)^i (\gamma_1|\cdots|\gamma_i\gamma_{i+1}|\cdots|\gamma_k) + (-1)^k (\gamma_1|\cdots|\gamma_{k-1}).$$

A relative fundamental class $F \in C_2(\Gamma)$ is defined by the property

$$\partial F = \sum_{i=1}^n (\gamma_i).$$

Let $[\Gamma, \partial\Gamma] = \sum_{i=2}^n (\gamma_1 \cdots \gamma_{i-1}|\gamma_i) \in C_2(\Gamma)$, then

Lemma 6.2. $[\Gamma, \partial\Gamma]$ is a relative fundamental class.

Proof: The proof is left to the reader.

We will now give the symplectic form A in terms of group cohomology. We denote by \cup_b the cup product of Eilenberg-MacLane cochains using the form b on the coefficients.

Proposition 6.3.

$$A(\alpha, \alpha') = \sum_{i=1}^n \langle c \cup_b x'_i, (\gamma_i) \rangle - \langle c \cup_b c', [\Gamma, \partial\Gamma] \rangle$$

We will use the next Lemmas to prove Proposition 6.3.

Lemma 6.4.

$$\int_{e_i} B(f, \tilde{\alpha}') + \int_{\hat{e}_i} B(f, \tilde{\alpha}') = b(c(\gamma_i), f'(v_i^\infty)) - b(c(\gamma_i), f'(v_i))$$

Proof: Recall $\gamma_i \widehat{e}_i = -e_i$, so that $\widehat{e}_i = -\gamma_i^{-1} e_i$. We then have

$$\begin{aligned}
\int_{e_i} B(f, \widetilde{\alpha}') + \int_{\widehat{e}_i} B(f, \widetilde{\alpha}') &= \int_{e_i} B(f, \widetilde{\alpha}') + \int_{\widehat{e}_i} B(f, \widetilde{\alpha}') \\
&= \int_{e_i} B(f, \widetilde{\alpha}') + \int_{\gamma_i^{-1} e_i} B(f, \widetilde{\alpha}') \\
&= \int_{e_i} B(f, \widetilde{\alpha}') + \int_{e_i} (\gamma_i^{-1})^* B(f, \widetilde{\alpha}') \\
&= \int_{e_i} B(f, \widetilde{\alpha}') + \int_{e_i} B((\gamma_i^{-1})^* f, (\gamma_i^{-1})^* \widetilde{\alpha}') \\
&= \int_{e_i} B(f, \widetilde{\alpha}') + \int_{e_i} B(Ad_{\rho(\gamma_i)}(\gamma_i^{-1})^* f, Ad_{\rho(\gamma_i)}(\gamma_i^{-1})^* \widetilde{\alpha}') \\
&= \int_{e_i} B(f - Ad_{\rho(\gamma_i)}(\gamma_i^{-1})^* f, \widetilde{\alpha}') \\
&= \int_{e_i} B(c(\gamma_i), \widetilde{\alpha}') \\
&= b(c(\gamma_i), f'(v_i^\infty)) - b(c(\gamma_i), f'(v_i))
\end{aligned}$$

□

Lemma 6.5.

$$\sum_{i=1}^n b(c(\gamma_i), f'(v_i)) = \sum_{i=1}^n b(c(\gamma_i), f'(v_i^\infty)) - \sum_{i=1}^n \langle c \cup_b y_i, (\gamma_i) \rangle + \langle c \cup_b c', [\Gamma, \partial\Gamma] \rangle$$

Proof: By definition, for any $x \in \mathbb{H}^2$ and $\gamma \in \Gamma$ we have

$$c'(\gamma) = f'(x) - Ad_{\rho(\gamma)} f'(\gamma^{-1} x)$$

Let $\gamma = \gamma_i$ and $x = v_i$, then

$$c'(\gamma_i) = f'(v_i) - Ad_{\rho(\gamma_i)} f'(v_{i+1})$$

Using $f'(v_1) = 0$, we obtain

$$\begin{aligned}
c'(\gamma_1 \cdots \gamma_i) &= f'(v_1) - Ad_{\rho(\gamma_1 \cdots \gamma_i)} f'(\gamma_i^{-1} \cdots \gamma_1^{-1} v_1) \\
&= -Ad_{\rho(\gamma_1 \cdots \gamma_i)} f'(v_{i+1}).
\end{aligned}$$

We will also need

$$\begin{aligned}
c'(\gamma_1 \cdots \gamma_i) &= c'(\gamma_1 \cdots \gamma_{i-1}) + Ad_{\rho(\gamma_1 \cdots \gamma_{i-1})} c'(\gamma_i) \\
&= c'(\gamma_1) + Ad_{\rho(\gamma_1)} c'(\gamma_2) + \cdots + Ad_{\rho(\gamma_1 \cdots \gamma_{i-1})} c'(\gamma_i)
\end{aligned}$$

and, since $\gamma_1 \cdots \gamma_n = 1$,

$$0 = c'(\gamma_1 \cdots \gamma_n) = c'(\gamma_1) + Ad_{\rho(\gamma_1)} c'(\gamma_2) + \cdots + Ad_{\rho(\gamma_1 \cdots \gamma_{n-1})} c'(\gamma_n)$$

We then have,

$$\begin{aligned}
\sum_{i=1}^n b(c(\gamma_i), f'(v_i)) &= - \sum_{i=1}^n b(c(\gamma_i), Ad_{\rho(\gamma_1 \cdots \gamma_i)^{-1}} c'(\gamma_1 \cdots \gamma_{i-1})) \\
&= - \sum_{i=1}^n b(Ad_{\rho(\gamma_1 \cdots \gamma_{i-1})} c(\gamma_i), c'(\gamma_1) + Ad_{\rho(\gamma_1)} c'(\gamma_2) + \cdots + Ad_{\rho(\gamma_1 \cdots \gamma_{i-2})} c'(\gamma_{i-1})) \\
&= - \sum_{i=1}^n \sum_{j=1}^{i-1} b(Ad_{\rho(\gamma_1 \cdots \gamma_{i-1})} c(\gamma_i), Ad_{\rho(\gamma_1 \cdots \gamma_{j-1})} c'(\gamma_j)) \\
&= - \sum_{j=1}^n \sum_{i=j+1}^n b(Ad_{\rho(\gamma_1 \cdots \gamma_{i-1})} c(\gamma_i), Ad_{\rho(\gamma_1 \cdots \gamma_{j-1})} c'(\gamma_j)) \\
&= \sum_{j=1}^n \sum_{i=1}^j b(Ad_{\rho(\gamma_1 \cdots \gamma_{i-1})} c(\gamma_i), Ad_{\rho(\gamma_1 \cdots \gamma_{j-1})} c'(\gamma_j)) \\
&= \sum_{j=1}^n b(c(\gamma_1 \cdots \gamma_j), Ad_{\rho(\gamma_1 \cdots \gamma_{j-1})} c'(\gamma_j)) \\
&= \sum_{j=1}^n b(c(\gamma_1 \cdots \gamma_{j-1}) + Ad_{\rho(\gamma_1 \cdots \gamma_{j-1})} c(\gamma_j), Ad_{\rho(\gamma_1 \cdots \gamma_{j-1})} c'(\gamma_j)) \\
&= \sum_{j=1}^n b(c(\gamma_1 \cdots \gamma_{j-1}), Ad_{\rho(\gamma_1 \cdots \gamma_{j-1})} c'(\gamma_j)) + \sum_{j=1}^n b(c(\gamma_j), c'(\gamma_j)) \\
&= \langle c \cup_b c', [\Gamma, \partial\Gamma] \rangle + \sum_{j=1}^n b(c(\gamma_j), f'(v_j^\infty) - Ad_{\rho(\gamma_j)} f'(v_j^\infty)) \\
&= \langle c \cup_b c', [\Gamma, \partial\Gamma] \rangle + \sum_{j=1}^n b(c(\gamma_j), f'(v_j^\infty)) - \sum_{j=1}^n \langle B(c, y'_j), (\gamma_j) \rangle
\end{aligned}$$

□

Proof of Proposition 6.3:

$$\begin{aligned}
A(\alpha, \alpha') &= \int_{\Sigma} B(\alpha, \alpha') \\
&= \int_{\mathcal{D}} B(\tilde{\alpha}, \tilde{\alpha}') \\
&= \int_{\partial\mathcal{D}} B(\tilde{\alpha}, f') \\
&= \sum_{i=1}^n \left(\int_{e_i} B(\tilde{\alpha}, f') + \int_{\widehat{e}_i} B(\tilde{\alpha}, f') \right) \\
&= \sum_{j=1}^n \langle c \cup_b x'_j, (\gamma_j) \rangle - \langle c \cup_b c', [\Gamma, \partial\Gamma] \rangle
\end{aligned}$$

□

6.3. Correspondence between $M_r(\mathbb{S}^3)$ and $\text{Hom}(\Gamma, T; SU(2)) / SU(2)$. We now restrict to the case $G = SU(2)$. We define the isomorphism

$$\Upsilon : \text{Hom}(\Gamma, T; SU(2)) \rightarrow \widetilde{M}_r,$$

where \widetilde{M}_r is the closed polygonal linkages in \mathbb{S}^3 based at a point, by

$$\Upsilon(\rho) = (\rho(\gamma_1), \dots, \rho(\gamma_n)).$$

This induces an isomorphism, which we also denote by Υ ,

$$\Upsilon : \text{Hom}(\Gamma, T; SU(2)) / SU(2) \rightarrow M_r.$$

The differential $d\Upsilon_\rho : T_\rho(\text{Hom}(\Gamma, T; SU(2)) / SU(2)) \rightarrow T_{\Upsilon(\rho)} M_r$ is then defined by

$$d\Upsilon_\rho(c) = (dR_{\rho(\gamma_1)}c(\gamma_1), \dots, dR_{\rho(\gamma_n)}c(\gamma_n)).$$

Here $T_\rho(\text{Hom}(\Gamma, T; SU(2)) / SU(2))$ is identified with an element of $\mathbb{Z}_{par}^1(\Gamma, \mathfrak{g})$. We have

$$d\Upsilon_\rho(c) = (dR_{g_1}x_1 - dL_{g_1}x_1, \dots, dR_{g_n}x_n - dL_{g_n}x_n)$$

and

$$d\Upsilon_\rho(c') = (dR_{g_1}x'_1 - dL_{g_1}x'_1, \dots, dR_{g_n}x'_n - dL_{g_n}x'_n).$$

Recall, the symplectic form on M_r is given by

$$\tilde{\omega} = \sum_{i=1}^n \omega_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=i+1}^n (Ad_{g_1 \dots g_{i-1}} \bar{\theta}_i \wedge_b Ad_{g_1 \dots g_{j-1}} \bar{\theta}_j).$$

We can now prove the main result of this section

Theorem 6.6. $\Upsilon^* \tilde{\omega} = A$

Proof:

First we note that

$$\Upsilon^* \bar{\theta}_i(c) = c(\gamma_i)$$

and

$$\begin{aligned} (\Upsilon^* \omega_i)(c, c') &= \omega_i(dR_{g_i}c(\gamma_i), dR_{g_i}c'(\gamma_i)) \\ &= -\frac{1}{2} (Ad_{g_i^{-1}}c(\gamma_i) + c(\gamma_i), x'_i) \\ &= -\frac{1}{2} (c(\gamma_i), Ad_{g_i}x'_i + x'_i) \\ &= -\frac{1}{2} (c(\gamma_i), c'(\gamma_i)) - (c(\gamma_i), Ad_{g_i}x'_i) \\ &= -\frac{1}{2} (Ad_{g_1 \dots g_{i-1}}c(\gamma_i), Ad_{g_1 \dots g_{i-1}}c'(\gamma_i)) + \langle c \cup_b x'_i, (\gamma_i) \rangle \end{aligned}$$

It follows that

$$\begin{aligned}
(\Upsilon^* \tilde{\omega})(c, c') &= \sum_{i=1}^n (\Upsilon^* \omega_i)(c, c') + \frac{1}{2} \sum_{i=1}^n \sum_{j=i+1}^n \Upsilon^* (Ad_{g_1 \dots g_{i-1}} \bar{\theta}_i \wedge_b Ad_{g_1 \dots g_{j-1}} \bar{\theta}_j) (c, c') \\
&= \sum_{i=1}^n \langle c \cup_b x'_i, (\gamma_i) \rangle - \sum_{i=1}^n \frac{1}{2} (Ad_{g_1 \dots g_{i-1}} c(\gamma_i), Ad_{g_1 \dots g_{i-1}} c'(\gamma_i)) \\
&\quad + \sum_{i=1}^n \sum_{j=i+1}^n (Ad_{g_1 \dots g_{i-1}} c(\gamma_i), Ad_{g_1 \dots g_{j-1}} c'(\gamma_j)) \\
&\quad - \sum_{i=1}^n \sum_{j=i+1}^n (Ad_{g_1 \dots g_{i-1}} c'(\gamma_i), Ad_{g_1 \dots g_{j-1}} c(\gamma_j)) \\
&= \sum_{i=1}^n \langle c \cup_b x'_i, (\gamma_i) \rangle - \sum_{i=1}^n \frac{1}{2} (Ad_{g_1 \dots g_{i-1}} c(\gamma_i), Ad_{g_1 \dots g_{i-1}} c'(\gamma_i)) \\
&\quad + \sum_{j=2}^n \sum_{i=1}^{j-1} (Ad_{g_1 \dots g_{i-1}} c(\gamma_i), Ad_{g_1 \dots g_{j-1}} c'(\gamma_j)) \\
&\quad + \sum_{i=1}^n \sum_{j=1}^i (Ad_{g_1 \dots g_{i-1}} c'(\gamma_i), Ad_{g_1 \dots g_{j-1}} c(\gamma_j)) \\
&= \sum_{i=1}^n \langle c \cup_b x'_i, (\gamma_i) \rangle + \sum_{j=2}^n \sum_{i=1}^{j-1} (Ad_{g_1 \dots g_{i-1}} c(\gamma_i), Ad_{g_1 \dots g_{j-1}} c'(\gamma_j)) \\
&= \sum_{i=1}^n \langle c \cup_b x'_i, (\gamma_i) \rangle + \sum_{j=2}^n (Ad_{g_1 \dots g_{i-1}} c'(\gamma_i), c(\gamma_1 \dots \gamma_{i-1})) \\
&= \sum_{i=1}^n \langle c \cup_b x'_i, (\gamma_i) \rangle - \langle c \cup_b c', [\Gamma, \partial \Gamma] \rangle \\
&= A(\alpha, \alpha')
\end{aligned}$$

□

It is easily seen that the functions ℓ_i from §4.2 corresponds to the following Goldman functions. Let $\varphi : G \rightarrow \mathbb{R}$ be defined by $\varphi(g) = \cos^{-1}(-\frac{1}{2}\text{trace}(g))$. We then defined the function $\varphi_\gamma : \text{Hom}(\Gamma, T; SU(2)) / SU(2) \rightarrow \mathbb{R}$ by $\varphi_\gamma a(\rho) = \varphi(\rho(-ga))$. We see that

$$\Upsilon^* \ell_i = \varphi_{\gamma_1 \dots \gamma_i}$$

Then choosing an maximal collection of nonintersecting diagonal on M_r corresponds to a pair of pants decomposition on Σ .

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